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RADON TRANSFORMS ON AFFINE GRASSMANNIANS

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Dedicated to Professor Lawrence Zalcman on the occasion of his 60th birthday

ABSTRACT. We develop an analytic approach to the Radon transform $\hat{f}(\zeta) = \int_{\tau \subset \zeta} f(\tau)$, where $f(\tau)$ is a function on the affine Grassmann manifold $G(n,k)$ of k-dimensional planes in $\mathbb{R}^n$, and ζ is a k'-dimensional plane in the similar manifold $G(n,k')$, $k' > k$. For $f \in L^p(G(n,k))$, we prove that this transform is finite almost everywhere on $G(n,k')$ if and only if $1 \le p < (n-k)/(k'-k)$, and obtain explicit inversion formulas. We establish correspondence between Radon transforms on affine Grassmann manifolds and similar transforms on standard Grassmann manifolds of linear subspaces of $\mathbb{R}^{n+1}$. It is proved that the dual Radon transform can be explicitly inverted for $k + k' \ge n-1$, and interpreted as a direct, "quasi-orthogonal" Radon transform for another pair of affine Grassmannians. As a consequence we obtain that the Radon transform and the dual Radon transform are injective simultaneously if and only if $k + k' = n-1$. The investigation is carried out for locally integrable and continuous functions satisfying natural weak conditions at infinity.

1. INTRODUCTION

In this paper we investigate integral Radon transforms of functions defined on manifolds of planes in the euclidean space $\mathbb{R}^n$. Let $G(n,k)$ and $G(n,k')$ be a pair of the affine Grassmann manifolds of k-dimensional and k'-dimensional planes in $\mathbb{R}^n$, respectively. We suppose $0 \le k < k' < n$. The case $k = 0$ corresponds to points in $\mathbb{R}^n$. Given sufficiently good functions $f(\tau)$ on $G(n,k)$ and $\varphi(\zeta)$ on $G(n,k')$, we consider the following integral transforms:

$$\hat{f}(\zeta) = \int\limits_{\tau \subset \zeta} f(\tau), \qquad \check{\varphi}(\tau) = \int\limits_{\zeta \supset \tau} \varphi(\zeta), \tag{1.1}$$

the integration being performed against the corresponding canonical measures. The first integral is called *the Radon transform* of f and denotes integration over all k-planes τ in the k'-plane ζ. The second one is called *the dual Radon transform* of φ and integrates over all k'-planes ζ containing the k-plane τ. Our first goal in this paper is to find possibly maximal subclasses of locally integrable functions, like L^p

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spaces or weighted L^1 spaces, on which transformations (1.1) are well defined and injective. The second goal is to obtain explicit inversion formulas which allow us to recover f and φ from $\hat{f}$ and $\check{\varphi}$, respectively.

These problems agree with the general set-up due to I. M. Gelfand [Ge]. Numerous particular cases and their modifications are known and have a long history [H2], [Ru4]. The similar problems for standard Grassmann manifolds $G_{n,k}$, $G_{n,k'}$ of k-dimensional and k'-dimensional linear subspaces of $\mathbb{R}^n$ were studied in [GR], where one can find further references to works of S. Helgason, I. M. Gelfand, M. I. Graev, Z. Ya. Shapiro, S. G. Gindikin, E. E. Petrov, E. L. Grinberg, F. Gonzalez, T. Kakehi, and others.

From the point of view of analysis, there is an important difference between our problem and the similar one for standard Grassmannians. Namely, in contrast to $G_{n,k}$, the manifold $G(n,k)$ of affine planes is noncompact, and one should specify behavior of functions at infinity. For infinitely differentiable and rapidly decreasing functions, the Radon transforms on affine Grassmannians were studied by M. I. Graev, F. Gonzalez and T. Kakehi.

M. I. Graev [Gr] identifies planes in $\mathbb{R}^n$ with matrices, the entries of which serve as local coordinates on $G(n,k)$. Using this identification, the relevant Fourier analysis and differential forms, he obtained an inversion formula for $\hat{f}$ in the case $k'-k$ even. This is the so-called local case, when in order to recover f at the "point" τ one only needs $\hat{f}(\zeta)$ for ζ infinitesimally close to τ. Another inversion formula in the same case $k'-k$ even was obtained by F. Gonzalez and T. Kakehi [GK1] in the form $f = D(\hat{f})^\vee$. Here D is a certain Pfaffian-type differential operator on $G(n,k)$ which is defined in the corresponding Lie algebra language. Such differential operators were introduced in [GK1] and [GK2] to characterize ranges of transformations (1.1). One should also mention the paper by Strichartz [Str], who developed L^2 harmonic analysis on Grassmannian bundles.

The cited papers leave open the following important questions: (a) For which locally integrable functions f is the Radon transform $\hat{f}$ well defined; (b) How do we reconstruct f from $\hat{f}$ (explicitly and pointwise) under natural minimal assumptions for f? The "nonlocal" case $k'-k$ odd is of particular interest, because in this case, no simple/readable inversion formulas are available, even for "good" f; see [Gr] for comments on this subject.

Much less is known about inversion of the dual transform $\varphi \to \check{\varphi}$. For $k = 0$, $k' = n-1$, the following results can be found in the literature. The case $n = 2$ was studied by Radon [Rad, Section B] in 1917. He considered differentiable functions, radial derivatives of which are integrable with the logarithmic weight. An inversion formula for the dual Radon transform for all $n \geq 2$ was obtained in 1964 by Helgason [H1], [H2, pp. 17, 51] in the framework of Schwartz functions orthogonal to all polynomials. For arbitrary Schwartz functions, such a formula was obtained by Gonzalez [Go] (n odd) and Solmon [So2] (arbitrary n). By making use of the Fourier transform technique, Madych and Solmon [MS, Corollary 2 on p. 84] justified the inversion formula under weaker smoothness and decay assumptions. See also [He] for the distributional approach.

For $k > 0$, characterization of the range of the dual Radon transform was obtained in [GK2] for functions $\varphi \in C^\infty$ provided $k + k' \geq n$. Of course, the dual Radon transform $\check{\varphi}$ is well defined for arbitrary locally integrable function, but we do not know how to invert it in such a general setup, rather than in the sense of distributions. In the present paper we obtain new inversion results under assumptions that agree with [Rad], and much weaker than those in [So2] and [MS].

Main results. Our approach is purely analytical and different from those mentioned above. It enables us to investigate both "even" and "odd" cases, and get rid of unnecessary restrictions. Moreover, it allows us to get for free a number of important qualitative results by deriving them from those for compact Grassmannians.

Given a k-plane $\tau \in G(n,k)$, we denote by $|\tau|$ the euclidean distance of τ to the origin of $\mathbb{R}^n$.

Proposition 1.1. *If $f \in L^p(G(n,k))$, $1 \le p < (n-k)/(k'-k)$, then $\hat f(\zeta)$ is finite for almost all $\zeta \in G(n,k')$. If $p \ge (n-k)/(k'-k)$ and*

$$f(\tau) = (2+|\tau|)^{(k-n)/p}(\log(2+|\tau|))^{-1} \ (\in L^p),$$

then $\hat f(\zeta) \equiv \infty$. In particular, if f is a continuous function satisfying $f(\tau) = O(|\tau|^{-\lambda})$, then $\hat f(\zeta)$ is finite for all $\zeta \in G(n,k')$ provided $\lambda > k'-k$, and can be identically infinite if $\lambda \le k'-k$.

Instead of L^p-spaces we also consider weighted L^1 spaces

$$L^1_\lambda(G(n,k)) = \left\{ f(\tau) \, : ||f|| = \int\limits_{G(n,k)} \frac{|f(\tau)| \, d\tau}{(1+|\tau|)^\lambda} < \infty \right\}. \tag{1.2}$$

These consist of locally integrable functions having prescribed behavior at infinity.

Proposition 1.2. *For $\lambda = n-k'$, the Radon transform $f \to \hat f$ is a linear bounded operator from $L^1_\lambda(G(n,k))$ to $L^1_{\lambda+\delta}(G(n,k'))$, $\quad \forall \delta > 0$.*

The exponent $\lambda = n-k'$ is best possible. For any $\varepsilon > 0$, there exists $f \in L^1_{n-k'+\varepsilon}$ such that $\hat f(\zeta) \equiv \infty$. By Hölder's inequality, the space $L^p(G(n,k))$, $1 \le p < (n-k)/(k'-k)$, continuously embeds in $L^1_{n-k'}(G(n,k))$.

In order to obtain explicit inversion formulas for (1.1), we reduce the problem to the similar one for integrable functions on ordinary (compact) Grassmannians, and make use of our previous results from [GR]. Given a k-plane $\tau \in G(n,k)$, we denote by $\mu(\tau) \in G_{n+1,k+1}$ the smallest subspace of $\mathbb{R}^{n+1}$ containing the "lifted" plane $\tau + e_{n+1}$, e_{n+1} being the coordinate unit vector of the additional x_{n+1}-axis. For $\zeta \in G(n,k')$, we denote by $\mu_\perp(\zeta) \in G_{n+1,n-k'}$ the orthogonal complement of $\mu(\zeta)$ in $\mathbb{R}^{n+1}$. We establish correspondence between transforms (1.1) on affine Grassmannians and similar transforms on ordinary Grassmannians, and prove the following

Proposition 1.3. *For $f \in L^p(G(n,k))$, $1 \le p < (n-k)/(k'-k)$, or $f \in L^1_{n-k'}(G(n,k))$, the Radon transform $f(\tau) \to \hat f(\zeta)$ is injective if and only if $k+k' \le n-1$. Under this condition, the function $f(\tau)$ can be recovered by the formula*

$$f(\tau) = (1+|\tau|^2)^{-(k'+1)/2}(R^{-1}\Psi)(\mu(\tau)), \tag{1.3}$$

where

$$\Psi(\mu(\zeta)) = \frac{\sigma_k}{\sigma_{k'}}(1+|\zeta|^2)^{(k+1)/2}\hat f(\zeta),$$

σ_k and $\sigma_{k'}$ denote the areas of unit spheres (of dimensions k and k', respectively), and R^{-1} is the inverse Radon transform on the Grassmann manifold $G_{n+1,k+1}$.

Explicit formulas for R^{-1} were obtained in [GR] and presented in Section 4; see formulas (4.7) and (4.8).

The similar result for the dual Radon transform reads as follows.

Proposition 1.4. *For $\varphi \in L^1_{k+1}(G(n,k'))$, the dual Radon transform $\varphi(\zeta) \to \check{\varphi}(\tau)$ is injective if and only if $k+k' \ge n-1$. Under this condition, the function $\varphi(\zeta)$ can be recovered by the formula*

$$\varphi(\zeta) = (1+|\zeta|^2)^{(k-n)/2}(R^{-1}\Psi_\perp)(\mu_\perp(\zeta)), \tag{1.4}$$

where

$$\Psi_\perp(\mu_\perp(\tau)) = (1+|\tau|^2)^{(n-k')/2}\check{\varphi}(\tau),$$

the operator R^{-1} being defined by the formulas (4.7) *and* (4.8) *in which dimensions $k+1$ and $k'+1$ should be replaced by $n-k$ and $n-k'$, respectively.*

Propositions 1.3 and 1.4 include continuous functions with suitable behavior at infinity; see Theorems 4.2 and 4.4 for details.

Corollary 1.5. *Let $0 \le k < k' \le n-1$. The Radon transform $f(\tau) \to \hat{f}(\zeta)$ defined on $f \in L^1_{n-k'}(G(n,k))$ and acting from $G(n,k)$ to $G(n,k')$, and the dual Radon transform $\varphi(\zeta) \to \check{\varphi}(\tau)$ defined on $\varphi \in L^1_{k+1}(G(n,k'))$ and acting from $G(n,k')$ to $G(n,k)$, are injective simultaneously if and only if $k+k' = n-1$.*

Being applied to the case $k=0$, the assumption $\varphi \in L^1_{k+1}(G(n,k'))$ is much weaker than that in [So2], where φ is a Schwartz function. It is interesting to note that for $k+k' > n-1$, our Proposition 1.4 allows increasing functions of order $O(|\zeta|^\lambda)$, $\lambda < k'+k+1-n$. In the case $k+k'=n-1$ of simultaneous injectivity we have to restrict to slowly decreasing functions. This is consistent with the logarithmic assumption of Radon [Rad] but does not assume any smoothness.

Since the Radon transform and its dual on ordinary Grassmannians can be expressed one through another, it would be natural to find a similar "one-through-another" representation on affine Grassmannians. We show (see Theorem 5.5) that the dual Radon transform $\check{\varphi}(\tau)$ can be represented as a usual, "quasi-orthogonal" Radon transform for the corresponding pair of affine Grassmannians. Such a representation realizes by the map

$$\begin{gathered} G^0(n,k) \ni \tau \xrightarrow{\ \nu\ } t = \nu(\tau) \in G(n, n-k-1), \\ G^0(n,k) = \{\tau \in G(n,k) \,:\, 0 \notin \tau\}; \end{gathered} \tag{1.5}$$

see Definition 5.1. In the simplest case $k=0$, when $\tau = x \in \mathbb{R}^n \setminus \{0\}$, $\nu(x)$ is the hyperplane orthogonal to the vector x and passing through the point $-x/|x|^2$. We call (1.5) a *quasi-orthogonal inversion transformation* from $G(n,k)$ to $G(n,n-k-1)$.

This paper is organized as follows. In Section 2 we give basic definitions and investigate operators (1.1) on radial functions. These depend only on the distance of the plane to the origin. We show that operators (1.1) on such functions are represented by one-dimensional Abel-type integrals. Inversion of those is standard [Ru1], and we omit it. By using these representations and duality between operators (1.1), we prove Propositions 1.1 and 1.2. In Section 3 we establish correspondence between transforms (1.1) and similar transforms on ordinary Grassmannians. Section 4 contains auxiliary facts from [GR] and justification of Propositions 1.3, 1.4. Section 5 is devoted to the quasi-orthogonal inversion transformation (1.5) and the corresponding connection between the Radon transform and its dual. For $k=0$, this connection was established by Quinto for L^2 functions; see [Q], formula (3.4) on p. 413.

This paper is essentially self-contained up to implementation of main results from [GR] and minor technicalities from [Ru2]. Useful information about the case $k = 0$, related to Radon transforms on $\mathbb{R}^n$, can be found in [Ru3].

2. Some properties of Radon transforms

2.1. **Basic definitions.** Let $G(n,k)$ be the affine Grassmann manifold of all non-oriented k-dimensional planes in $\mathbb{R}^n, 0 \le k < n$. We denote by $G_{n,k}$ the standard Grassmann manifold of all k-dimensional linear subspaces of $\mathbb{R}^n$. Each subspace $\xi \in G_{n,k}$ represents a k-plane passing through the origin. Each plane $\tau \in G(n,k)$ is parameterized by the pair (ξ, u), where $\xi \in G_{n,k}$ and $u \in \xi^\perp$, the orthogonal complement to ξ in $\mathbb{R}^n$. We denote by $|\tau|$ the euclidean distance of $\tau = (\xi, u) \in G(n,k)$ to the origin of $\mathbb{R}^n$. Clearly, $|\tau| = |u|$ (the euclidean norm of u). The manifold $G(n,k)$ will be endowed with the product measure $d\tau = d\xi du$, where $d\xi$ is the $SO(n)$-invariant measure on $G_{n,k}$ of total mass 1, and du denotes the usual volume element on $\xi^\perp$. The group $\mathbf{M}(n)$ of isometries of $\mathbb{R}^n$ acts on $G(n,k)$ transitively.

For $k' > k$ and $\eta \in G_{n,k'}$, we denote by $G_k(\eta)$ the Grassmann manifold of all k-dimensional linear subspaces of η. In the following $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere S^{n-1} in $\mathbb{R}^n$; $e_1, \dots, e_n$ are coordinate unit vectors. Given $1 \le k < k' < n$, we use the following notation for coordinate planes:

$$\mathbb{R}^k = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_k, \qquad \mathbb{R}^{k'} = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_{k'},$$
$$\mathbb{R}^{k'-k} = \mathbb{R}e_{k+1} \oplus \ldots \oplus \mathbb{R}e_{k'}, \qquad \mathbb{R}^{n-k} = \mathbb{R}e_{k+1} \oplus \ldots \oplus \mathbb{R}e_n.$$

If τ_0 is a plane in $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}e_{n+1}$, and S^n is the unit sphere in $\mathbb{R}^{n+1}$, then $d(e_{n+1}, \tau_0)$ denotes the geodesic distance (on S^n) between the north pole e_{n+1} and the totally geodesic submanifold $S^n \cap \tau_0$.

The letter c stands for a constant that can be different at each occurrence. Given a real-valued expression A, we set $(A)_+^\lambda = A^\lambda$ if $A > 0$ and 0 if $A \le 0$. More notation will be introduced in due course.

2.2. **The Radon transform and the dual Radon transform.** Let $G(n,k)$ and $G(n,k')$ be a pair of affine Grassmann manifolds of k-planes τ and k'-planes ζ in $\mathbb{R}^n$, respectively, $1 \le k < k' \le n-1$. We write

$$\tau = (\xi, u) \in G(n,k), \quad u \in \xi^\perp; \quad \zeta = (\eta, v) \in G(n,k'), \quad v \in \eta^\perp. \tag{2.1}$$

The Radon transform of a function $f(\tau)$ on $G(n,k)$ is a function $\hat{f}(\zeta)$ on $G(n,k')$ defined by

$$\hat{f}(\zeta) = \int\limits_{\tau \subset \zeta} f(\tau) = \int\limits_{\xi \subset \eta} d_\eta \xi \int\limits_{\xi^\perp \cap \eta} f(\xi, v + x) dx. \tag{2.2}$$

Here $d_\eta \xi$ denotes the normalized measure on the Grassmannian $G_k(\eta)$ of all k-dimensional linear subspaces of η. The right-hand side of (2.2) gives precise meaning to the integral $\int_{\tau \subset \zeta} f(\tau)$ which denotes integration over all k-planes τ in the k'-plane ζ. Assuming $g \in SO(n)$ to be a rotation such that

$$g : \mathbb{R}^{k'} \to \eta, \qquad g : e_{k'+1} \to v/|v|,$$

and denoting $f_g(\tau) = f(g\tau)$, one can write (2.2) as

$$\hat{f}(\eta, v) = \int\limits_{G_{k',k}} d\sigma \int\limits_{\sigma^\perp \cap \mathbb{R}^{k'}} f_g(\sigma, |v|e_{k'+1} + y)\, dy \tag{2.3}$$

or

$$\hat{f}(\eta, v) = \int\limits_{SO(k')} d\gamma \int\limits_{\mathbb{R}^{k'-k}} f_g(\gamma(\mathbb{R}^k + |v|e_{k'+1} + z))\, dz. \tag{2.4}$$

The dual Radon transform of a function $\varphi(\zeta) \equiv \varphi(\eta, v)$ on $G(n, k')$ is a function $\check{\varphi}(\tau)$ on $G_{n,k}$ defined by

$$\check{\varphi}(\tau) = \int\limits_{\zeta \supset \tau} \varphi(\zeta) = \int\limits_{\eta \supset \xi} \varphi(\eta + u) d_\xi \eta = \int\limits_{\eta \supset \xi} \varphi(\eta, \mathrm{Pr}_{\eta^\perp} u) d_\xi \eta. \tag{2.5}$$

Here $\mathrm{Pr}_{\eta^\perp} u$ denotes the orthogonal projection of u $(\in \xi^\perp)$ onto $\eta^\perp (\subset \xi^\perp)$, where $d_\xi \eta$ is the relevant normalized measure. This transform integrates $\varphi(\zeta)$ over all k'-planes ζ containing the k-plane τ. In order to give (2.5) precise meaning, we choose a rotation $g_\xi \in SO(n)$ so that $g_\xi \mathbb{R}^k = \xi$, and let $SO(n-k)$ be the subgroup of rotations in the coordinate plane $\mathbb{R}^{n-k}$. Then (2.5) means

$$\check{\varphi}(\tau) \equiv \check{\varphi}(\xi, u) = \int\limits_{SO(n-k)} \varphi(g_\xi \rho \mathbb{R}^{k'} + u)\, d\rho. \tag{2.6}$$

Lemma 2.1. *The equality*

$$\int\limits_{G(n,k')} \hat{f}(\zeta) \varphi(\zeta) d\zeta = \int\limits_{G(n,k)} f(\tau) \check{\varphi}(\tau) d\tau \tag{2.7}$$

holds provided the integral in either side exists for f and φ replaced by $|f|$ and $|\varphi|$, respectively.

Proof. This statement follows from the rather general fact for the double fibration [H2, p. 57]. For convenience of the reader and in order to check normalization of measures under consideration, we give a direct proof. Let $G = SO(n)$, and denote by I the left-hand side of (2.7). We have

$$\begin{aligned}
I &= \int\limits_{G(n,k')} \hat{f}(\zeta) \varphi(\zeta) d\zeta \\
&= \int\limits_G dg \int\limits_{\mathbb{R}^{n-k'}} \hat{f}(g(\mathbb{R}^{k'} + w)) \varphi(g(\mathbb{R}^{k'} + w)) dw \\
&= \int\limits_G dg \int\limits_{\mathbb{R}^{n-k'}} \varphi(g(\mathbb{R}^{k'} + w)) dw \int\limits_{SO(k')} d\gamma \int\limits_{\mathbb{R}^{k'-k}} f(g\gamma(\mathbb{R}^k + w + z)) dz \\
&= \int\limits_{SO(k')} d\gamma \int\limits_G dg \int\limits_{\mathbb{R}^{k'-k}} dz \int\limits_{\mathbb{R}^{n-k'}} \varphi(g\gamma(\mathbb{R}^{k'} + w)) f(g\gamma(\mathbb{R}^k + w + z)) dw \\
&= \int\limits_G dg \int\limits_{\mathbb{R}^{k'-k}} dz \int\limits_{\mathbb{R}^{n-k'}} \varphi(g(\mathbb{R}^{k'} + w)) f(g(\mathbb{R}^k + w + z)) dw.
\end{aligned}$$

Now we replace $\mathbb{R}^k$ by $\rho^{-1}\mathbb{R}^k$, $\rho \in SO(n-k)$, then integrate in ρ, and set $w+z=s$. We obtain

$$
\begin{aligned}
I &= \int\limits_{SO(n-k)} d\rho \int\limits_G dg \int\limits_{\mathbb{R}^{n-k}} \varphi(g(\mathbb{R}^{k'}+s))f(g(\rho^{-1}\mathbb{R}^k+s))ds \quad (s=\rho^{-1}y)\\
&= \int\limits_{SO(n-k)} d\rho \int\limits_G dg \int\limits_{\mathbb{R}^{n-k}} \varphi(g(\mathbb{R}^{k'}+\rho^{-1}y))f(g\rho^{-1}(\mathbb{R}^k+y))dy \quad (g=\lambda\rho)\\
&= \int\limits_G d\lambda \int\limits_{\mathbb{R}^{n-k}} f(\lambda(\mathbb{R}^k+y))dy \int\limits_{SO(n-k)} \varphi(\lambda(\rho\mathbb{R}^{k'}+y))d\rho\\
&\quad (\text{set } \lambda\mathbb{R}^k=\xi,\ \lambda y = u)\\
&= \int\limits_{G_{n,k}} d\xi \int\limits_{\xi^\perp} f(\xi+u)\check{\varphi}(\xi+u)du = \int\limits_{G(n,k)} f(\tau)\check{\varphi}(\tau)d\tau.
\end{aligned}
$$

□

Corollary 2.2. *If $f \in L^1(G(n,k))$, then*

$$
\int\limits_{G(n,k')} \hat{f}(\zeta)d\zeta = \int\limits_{G(n,k)} f(\tau)d\tau, \tag{2.8}
$$

and therefore $\hat{f}(\zeta)$ is finite for almost all $\zeta \in G(n,k')$.

2.3. **Radon transforms of radial functions.** A function $f(\tau)$, $\tau \in G(n,k)$, of the form $f \equiv f_0(|\tau|)$ is called radial.

Lemma 2.3. *For $\tau \in G(n,k)$ and $\zeta \in G(n,k')$, let $r=|\tau|$, $s=|\zeta|$. If $f(\tau)=f_0(r)$ and $\varphi(\zeta)=\varphi_0(s)$, then $\hat{f}(\zeta)$ and $\check{\varphi}(\tau)$ are represented by Abel-type integrals*

$$
\hat{f}(\zeta) = \sigma_{k'-k-1}\int\limits_s^\infty f_0(r)(r^2-s^2)^{(k'-k)/2-1}r\,dr, \tag{2.9}
$$

$$
\check{\varphi}(\tau) = \frac{\sigma_{k'-k-1}\,\sigma_{n-k'-1}}{\sigma_{n-k-1}\,r^{n-k-2}}\int\limits_0^r \varphi_0(s)(r^2-s^2)^{(k'-k)/2-1}s^{n-k'-1}\,ds, \tag{2.10}
$$

provided the corresponding integrals exist in the Lebesgue sense.

Proof. For $k=0$, when τ is a point in $\mathbb{R}^n$, these formulas are known; see, e.g., [Ru3]. For $k>0$, (2.4) yields

$$
\hat{f}(\zeta) = \int\limits_{\mathbb{R}^{k'-k}} f_0(|w+z|)dz = \sigma_{k'-k-1}\int\limits_0^\infty t^{k'-k-1}f_0(\sqrt{t^2+s^2})dt
$$

because $|w+z|^2=|w|^2+|z|^2$, $|z|=t$, $|w|=|\zeta|=s$. This gives (2.9). Furthermore, by (2.5), (2.6),

$$
\check{\varphi}(\tau) \equiv \check{\varphi}(\xi,u) = \int\limits_{SO(n-k)} \varphi_0(|\mathrm{Pr}_{g_\xi\rho\mathbb{R}^{n-k'}}u|)d\rho,
$$

where $g_\xi \in SO(n)$, $g_\xi \mathbb{R}^k = \xi$, and $u \in \xi^\perp$. By setting $u = g_\xi r\theta$ where $\theta \in \mathbb{R}^{n-k} \cap S^{n-1} = S^{n-k-1}$, we have

$$\check{\varphi}(\tau) = \int\limits_{SO(n-k)} \varphi_0(|\mathrm{Pr}_{\rho\mathbb{R}^{n-k'}} r\theta|) d\rho = \frac{1}{\sigma_{n-k-1}} \int\limits_{S^{n-k-1}} \varphi_0(|\mathrm{Pr}_{\mathbb{R}^{n-k'}} r\sigma|) d\sigma.$$

To transform this integral we pass to bi-spherical coordinates on S^{n-k-1} [VK, pp. 12, 22]. Namely, we set $\sigma = a\cos\psi + b\sin\psi$,

$$a \in S^{k'-k-1} \subset \mathbb{R}^{k'-k}, \qquad b \in S^{n-k'-1} \subset \mathbb{R}^{n-k'}, \qquad 0 < \psi < \pi/2,$$

so that $d\sigma = \sin^{n-k'-1}\psi \cos^{k'-k-1}\psi \, d\psi \, da \, db$, and obtain

$$\check{\varphi}(\tau) = \frac{\sigma_{k'-k-1}\,\sigma_{n-k'-1}}{\sigma_{n-k-1}} \int\limits_0^{\pi/2} \varphi_0(r\sin\psi)\, \sin^{n-k'-1}\psi \cos^{k'-k-1}\psi \, d\psi.$$

This coincides with (2.10). □

We observe that integrals (2.9) and (2.10) transform one into another (up to weights and constant multiples) if we replace r and s by their reciprocals. This means that the Radon transform can be regarded as a dual Radon transform and vice versa (at least on radial functions). We study this important phenomenon in full generality in Section 5.

Example 2.4. The following useful formulas can be obtained from (2.9), (2.10) by elementary calculations. For $Re\,\alpha > 0$ and $a > 0$,

$$|\tau|^{-\alpha-k'+k} \overset{\wedge}{\longrightarrow} \lambda_1 |\zeta|^{-\alpha}, \qquad \lambda_1 = \frac{\pi^{(k'-k)/2}\,\Gamma(\alpha/2)}{\Gamma((\alpha+k'-k)/2)}. \tag{2.11}$$

$$(1+|\tau|^2)^{-(\alpha+k'-k)/2} \overset{\wedge}{\longrightarrow} \lambda_2 (1+|\zeta|^2)^{-\alpha/2}, \qquad \lambda_2 = \lambda_1. \tag{2.12}$$

$$(a^2-|\tau|^2)_+^{\alpha-1} \overset{\wedge}{\longrightarrow} \lambda_3 (a^2-|\zeta|^2)_+^{\alpha+(k'-k)/2-1}, \qquad \lambda_3 = \frac{\pi^{(k'-k)/2}\,\Gamma(\alpha)}{\Gamma(\alpha+(k'-k)/2)}. \tag{2.13}$$

$$|\zeta|^{\alpha+k'-n} \overset{\vee}{\longrightarrow} \lambda_4 |\tau|^{\alpha+k'-n}, \qquad \lambda_4 = \frac{\Gamma(\alpha/2)\,\Gamma((n-k)/2)}{\Gamma((\alpha+k'-k)/2)\,\Gamma((n-k')/2)}. \tag{2.14}$$

$$\frac{(|\zeta|^2-a^2)_+^{\alpha-1}}{|\zeta|^{n-k'-2}} \overset{\vee}{\longrightarrow} \lambda_5 \frac{(|\tau|^2-a^2)_+^{\alpha+(k'-k)/2-1}}{|\tau|^{n-k-2}}, \qquad \lambda_5 = \frac{\pi^{(k'-k)/2}\,\sigma_{n-k'-1}\,\Gamma(\alpha)}{\sigma_{n-k-1}\,\Gamma(\alpha+(k'-k)/2)}. \tag{2.15}$$

$$\frac{|\zeta|^{\alpha+k'-n}}{(1+|\zeta|^2)^{(\alpha+k'-k)/2}} \overset{\vee}{\longrightarrow} \lambda_6 \frac{|\tau|^{\alpha+k'-n}}{(1+|\tau|^2)^{\alpha/2}}, \qquad \lambda_6 = \frac{\pi^{(k'-k)/2}\,\sigma_{n-k'-1}\,\Gamma(\alpha/2)}{\sigma_{n-k-1}\,\Gamma((\alpha+k'-k)/2)}. \tag{2.16}$$

The last equality is especially important, and we present its proof (all the rest are left to the reader). Let

$$\varphi(\zeta) = \frac{|\zeta|^{\alpha+k'-n}}{(1+|\zeta|^2)^{(\alpha+k'-k)/2}}, \qquad c = \frac{\sigma_{k'-k-1}\,\sigma_{n-k'-1}}{\sigma_{n-k-1}}.$$

Then (2.10) yields

$$\begin{aligned}
\check{\varphi}(\tau) &= \frac{c}{r^{n-k-2}} \int_0^r \frac{(r^2-s^2)^{(k'-k)/2-1}\, s^{\alpha-1}}{(1+s^2)^{(\alpha+k'-k)/2}}\, ds \\
&= \frac{c}{2r^{n-k-2}} \int_1^{1+r^2} \frac{(1+r^2-t)^{(k'-k)/2-1}\,(t-1)^{\alpha/2-1}}{t^{(\alpha+k'-k)/2}}\, dt \\
&= \frac{\pi^{(k'-k)/2}\,\sigma_{n-k'-1}\,\Gamma(\alpha/2)}{\sigma_{n-k-1}\,\Gamma((\alpha+k'-k)/2)}\, \frac{r^{\alpha+k'-n}}{(1+r^2)^{\alpha/2}}.
\end{aligned}$$

Combining (2.11)-(2.16) with the duality (2.7), we get the following equalities that give precise information about behavior of $\hat{f}(\zeta)$ and $\check{\varphi}(\tau)$, in particular, about possible singularities.

Theorem 2.5. *For* $Re\,\alpha > 0$ *and* $a > 0$*, the following formulas hold:*

$$\int_{G(n,k)} \check{\varphi}(\tau) \frac{d\tau}{|\tau|^{\alpha+k'-k}} = \lambda_1 \int_{G(n,k')} \varphi(\zeta) \frac{d\zeta}{|\zeta|^{\alpha}}, \tag{2.17}$$

$$\int_{G(n,k)} \check{\varphi}(\tau) \frac{d\tau}{(1+|\tau|^2)^{(\alpha+k'-k)/2}} = \lambda_2 \int_{G(n,k')} \varphi(\zeta) \frac{d\zeta}{(1+|\zeta|^2)^{\alpha/2}}, \tag{2.18}$$

$$\int_{|\tau|<a} \check{\varphi}(\tau)(a^2-|\tau|^2)^{\alpha-1}\, d\tau = \lambda_3 \int_{|\zeta|<a} \varphi(\zeta)(a^2-|\zeta|^2)^{\alpha+(k'-k)/2-1} d\zeta, \tag{2.19}$$

$$\int_{G(n,k')} \hat{f}(\zeta)|\zeta|^{\alpha+k'-n} d\zeta = \lambda_4 \int_{G(n,k)} f(\tau)|\tau|^{\alpha+k'-n} d\tau, \tag{2.20}$$

$$\int\limits_{|\zeta|>a} \hat{f}(\zeta)\frac{(|\zeta|^2-a^2)^{\alpha-1}}{|\zeta|^{n-k'-2}}\,d\zeta = \lambda_5 \int\limits_{|\tau|>a} f(\tau)\frac{(|\tau|^2-a^2)^{\alpha+(k'-k)/2-1}}{|\tau|^{n-k-2}}\,d\tau, \tag{2.21}$$

$$\int\limits_{G(n,k')} \hat{f}(\zeta)\frac{|\zeta|^{\alpha+k'-n}}{(1+|\zeta|^2)^{(\alpha+k'-k)/2}}d\zeta = \lambda_6 \int\limits_{G(n,k)} f(\tau)\frac{|\tau|^{\alpha+k'-n}}{(1+|\tau|^2)^{\alpha/2}}d\tau. \tag{2.22}$$

It is assumed that either side of the corresponding equality exists in the Lebesgue sense.

Corollary 2.6. *If $f \in L^p(G(n,k))$, $1 \le p < (n-k)/(k'-k)$, then $\hat{f}(\zeta)$ is finite for almost all $\zeta \in G(n,k')$. If $p \ge (n-k)/(k'-k)$ and*

$$f(\tau) = (2+|\tau|)^{(k-n)/p}(\log(2+|\tau|))^{-1} \ (\in L^p),$$

then $\hat{f}(\zeta) \equiv \infty$. In particular, if f is a continuous function satisfying $f(\tau) = O(|\tau|^{-\lambda})$, then $\hat{f}(\zeta)$ is finite for all $\zeta \in G(n,k')$ provided $\lambda > k'-k$, and can be identically infinite if $\lambda \le k'-k$.

Proof. By Hölder's inequality, the right-hand side of (2.22) does not exceed $A\lambda_6||f||_p$, where

$$A^{p'} = \int\limits_{G(n,k)} \frac{|\tau|^{(\alpha+k'-n)p'}}{(1+|\tau|^2)^{\alpha p'/2}}\,d\tau = \sigma_{n-k-1}\int\limits_0^\infty \frac{r^{(\alpha+k'-n)p'+n-k-1}}{(1+r^2)^{\alpha p'/2}}\,dr$$

$(1/p+1/p'=1)$. If

$$1 \le p < (n-k)/(k'-k), \qquad \alpha > k-k'+(n-k)/p,$$

then $A<\infty$, and the left-hand side of (2.22) is finite. It follows that the Radon transform $\hat{f}(\zeta)$ is finite for almost all $\zeta \in G(n,k')$. If

$$p \ge (n-k)/(k'-k), \quad f(\tau) = (2+|\tau|)^{(k-n)/p}(\log(2+|\tau|))^{-1} \ (\in L^p),$$

then the integral in (2.9) diverges, and therefore $\hat{f}(\zeta) \equiv \infty$. If f is continuous and $f(\tau) = O(|\tau|^{-\lambda})$, then, for each $\zeta \in G(n,k')$, the integral in (2.9) is dominated by

$$c\int\limits_{|\zeta|}^\infty (1+r)^{-\lambda}(r^2-|\zeta|^2)^{(k'-k)/2-1}r\,dr.$$

This is finite if $\lambda > k'-k$. In the case $\lambda \le k'-k$ one can use the same counter-example as before. □

Remark 2.7. The particular case $k=0$ of Corollary 2.6, corresponding to the k'-plane Radon transform on $\mathbb{R}^n$, is due to Solmon [So1]. His proof is different and based on the estimate

$$\int\limits_{G(n,k')} \frac{\hat{f}(\zeta)\, d\zeta}{(1+|\zeta|)^{n-k'+\delta}} \le c \int\limits_{\mathbb{R}^n} \frac{f(x)\, dx}{(1+|x|)^{n-k'}}, \quad \forall \delta > 0, \quad f \ge 0.$$

We generalize this inequality, make it more precise, and give a simpler proof.

Theorem 2.8. *For $f \ge 0$,*

$$\int\limits_{G(n,k')} \frac{\hat{f}(\zeta)\, d\zeta}{(1+|\zeta|)^{n-k'+\delta}} \le c \int\limits_{G(n,k)} f(\tau)\rho(\tau) d\tau, \tag{2.23}$$

where $c = c(n, k', k)$,

$$\rho(\tau)) = \begin{cases} (1+|\tau|)^{k'-n-\min(\delta,0)} & \text{if } \delta \ne 0, \\ (1+|\tau|)^{k'-n} \log(1+|\tau|) & \text{if } \delta = 0. \end{cases}$$

Proof. By (2.7), the left-hand side of (2.23) is

$$\int\limits_{G(n,k)} f(\tau)\check{\varphi}(\tau) d\tau, \qquad \varphi(\zeta) = \frac{1}{(1+|\zeta|)^{n-k'+\delta}}.$$

Owing to (2.10),

$$\begin{aligned} \check{\varphi}(\tau) &= c\, r^{k+2-n} \int_0^r \frac{s^{n-k'-1}(r^2-s^2)^{(k'-k)/2-1}}{(1+s)^{n-k'+\delta}}\, ds \\ &= c \int_0^1 \frac{t^{n-k'-1}(1-t^2)^{(k'-k)/2-1}}{(1+rt)^{n-k'+\delta}}\, dt, \qquad r = |\tau|. \end{aligned}$$

This integral is bounded on any finite interval $0 \le r \le A$. If r is big enough, say, $r > 2$, we write

$$\check{\varphi}(\tau) = c \Big(\int_0^{1/r} + \int_{1/r}^{1/2} + \int_{1/2}^{1} \Big)(...),$$

and estimate each summand. After simple computations, we obtain $\check{\varphi}(\tau) \le c\rho(\tau)$, and (2.23) follows. □

The case $\delta > 0$, $k = 0$ in (2.23) gives Solmon's result. Modification of (2.23) for more general weight functions of the form $|\zeta|^\alpha (1+|\zeta|)^\beta$, and the similar estimates for the dual Radon transform, can be obtained by the same reasoning; cf. Lemma 2.6 from [Ru3] for $k = 0$. We leave this exercise to the interested reader.

Following Theorem 2.8, we restrict our consideration by locally integrable functions belonging to weighted L^1 spaces with a suitable weight at infinity. Owing to Theorem 2.5, this class of functions can be essentially extended by including functions with nonintegrable singularities at the "origin" $\{\tau : \tau \supset 0\}$ and/or on the "spheres" $\{\tau : |t| = a\}$, $a > 0$, but we shall not consider this generalization. We denote

$$L^1_\lambda(G(n,k)) = \Big\{ f(\tau) \, : ||f|| = \int\limits_{G(n,k)} \frac{|f(\tau)|\, d\tau}{(1+|\tau|)^\lambda} < \infty \Big\}. \tag{2.24}$$

In this notation Theorem 2.8 reads as follows.

Corollary 2.9. *For $\lambda = n - k'$, the Radon transform $f \to \hat{f}$ is a linear bounded operator from $L^1_\lambda(G(n,k))$ to $L^1_{\lambda+\delta}(G(n,k'))$, $\quad \forall \delta > 0$.*

Remark 2.10. As in Corollary 2.6, one can readily see that the exponent $\lambda = n-k'$ is best possible. In other words, there exists $f \in L^1_{n-k'+\varepsilon}(G(n,k))$, $\varepsilon > 0$, such that $\hat{f}(\zeta) \equiv \infty$. Furthermore, by Hölder's inequality, the space $L^p(G(n,k))$, $1 \le p < (n-k)/(k'-k)$, continuously embeds in $L^1_{n-k'}(G(n,k))$.

3. Correspondence between Radon transforms on affine Grassmannians and compact Grassmannians

3.1. **Basic relations.** We shall regard the euclidean space $\mathbb{R}^n = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_n$ as a coordinate hyperplane in $\mathbb{R}^{n+1} = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_{n+1}$. Given a linear subspace V of $\mathbb{R}^{n+1}$ and a positive integer $k < \dim V$, we denote by $G_k(V)$ the Grassmann manifold of all k-dimensional linear subspaces of V. In particular, we have $G_k(\mathbb{R}^n) = G_{n,k}$, $G_{k+1}(\mathbb{R}^{n+1}) = G_{n+1,k+1}$. To each k-plane τ in $\mathbb{R}^n$ we associate a $(k+1)$-dimensional linear subspace τ_0 in $\mathbb{R}^{n+1}$ containing the "lifted" plane $\tau + e_{n+1}$. This leads to a map

$$G(n,k) \ni \tau \quad = \quad (\xi, u) \xrightarrow{\ \mu\ } \tau_0 = \mu(\tau) = \xi \oplus \mathbb{R}u_0 \in G_{n+1,k+1}, \tag{3.1}$$

$$u_0 \quad = \quad \frac{u + e_{n+1}}{|u + e_{n+1}|} = \frac{u + e_{n+1}}{\sqrt{1+|u|^2}} \in S^n, \tag{3.2}$$

S^n being the unit sphere in $\mathbb{R}^{n+1}$. If $\theta = d(e_{n+1}, \tau_0)$ is the geodesic distance (on S^n) between the north pole e_{n+1} and the k-dimensional totally geodesic submanifold $S^n \cap \tau_0$ of S^n, then $|\tau| = |u| = \tan\theta$.

Remark 3.1. The map $\mu : G(n,k) \to G_{n+1,k+1}$ is not one-to-one, but it becomes such if we change the definition as follows:

$$\mu : G(n,k) \to G_{n+1,k+1} \setminus G_{k+1}(\mathbb{R}^n) \stackrel{\text{def}}{=} G^0_{n+1,k+1}. \tag{3.3}$$

The subset $G_{k+1}(\mathbb{R}^n)$ of $G_{n+1,k+1}$ has measure zero.

The map (3.3) extends to a one-to-one correspondence between functions on $G(n,k)$ and $G_{n+1,k+1}$ so that

$$F(\tau_0) = f(\mu^{-1}(\tau_0)), \qquad f(\tau) = F(\mu(\tau)), \tag{3.4}$$

where $\tau_0 \in G^0_{n+1,k+1}$, $\tau \in G(n,k)$.

Lemma 3.2. *Let $\tau \in G(n,k)$, $\tau_0 = \mu(\tau) \in G_{n+1,k+1}$, $f(\tau) = F(\tau_0)$. Then*

$$\int\limits_{G(n,k)} \frac{f(\tau)\, d\tau}{(1+|\tau|^2)^{(n+1)/2}} = \frac{\sigma_n}{\sigma_k} \int\limits_{G_{n+1,k+1}} F(\tau_0)\, d\tau_0, \tag{3.5}$$

$$\int\limits_{G(n,k)} f(\tau)\, d\tau = \frac{\sigma_n}{\sigma_k} \int\limits_{G_{n+1,k+1}} \frac{F(\tau_0)\, d\tau_0}{\cos^{n+1} d(e_{n+1}, \tau_0)}. \tag{3.6}$$

Proof. Let us transform the integral $I = \int_{G_{n+1,k+1}} F(\tau_0)\, d\tau_0$. Owing to the formula (2.5) from [Ru2],

$$I = c \int_0^{\pi/2} \sin^{n-k-1}\omega\, \cos^k\omega\, d\omega \int\limits_{SO(n)} F(\gamma g_\omega^{-1} \mathbb{R}^{k+1}) d\gamma, \tag{3.7}$$

where g_ω denotes a rotation in the plane (e_{k+1}, e_{n+1}) so that

$$g_\omega e_{n+1} = e_{k+1}\cos\omega + e_{n+1}\sin\omega, \qquad d(g_\omega e_{n+1}, S^n \cap \mathbb{R}^{k+1}) = \omega;$$
$$c = \sigma_{k'-k-1}\sigma_k/\sigma_{k'}, \qquad \mathbb{R}^{k+1} = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_{k+1},$$

and $SO(n)$ is the group of rotations in the coordinate hyperplane $\mathbb{R}^n$. By taking into account that

$$g_\omega^{-1}\mathbb{R}^{k+1} = \mathbb{R}^k \oplus \mathbb{R}g_\omega^{-1}e_{k+1}, \qquad g_\omega^{-1}e_{k+1} = e_{k+1}\sin\omega + e_{n+1}\cos\omega,$$

the inner integral in (3.7) can be written as

$$\begin{aligned} &\int\limits_{SO(n)} F(\gamma(\mathbb{R}^k \oplus \mathbb{R}(e_{k+1}\sin\omega + e_{n+1}\cos\omega)))\,d\gamma \\ &\qquad (\text{replace } \gamma \text{ by } \gamma\lambda,\ \lambda \in SO(n-k), \text{ and integrate in } \lambda) \\ &= \int\limits_{SO(n)} d\gamma \int\limits_{SO(n-k)} F(\gamma(\mathbb{R}^k \oplus \mathbb{R}(\lambda e_{k+1}\sin\omega + e_{n+1}\cos\omega)))\,d\lambda. \end{aligned} \tag{3.8}$$

Here $SO(n-k)$ is the group of rotations in $\mathbb{R}^{n-k} = \mathbb{R}e_{k+1} \oplus \ldots \oplus \mathbb{R}e_n$. We plug (3.8) in (3.7) and change the variable $\tan\omega = s$. This gives

$$\begin{aligned} I &= \int\limits_0^\infty \frac{s^{n-k-1}\,ds}{(\sqrt{1+s^2})^{n+1}} \\ &\quad \times \int\limits_{SO(n)} d\gamma \int\limits_{SO(n-k)} F(\gamma(\mathbb{R}^k \oplus \mathbb{R}(\frac{s\lambda e_{k+1}}{\sqrt{1+s^2}} + \frac{e_{n+1}}{\sqrt{1+s^2}})))\,d\lambda \\ &= \frac{\sigma_k}{\sigma_n}\int\limits_{SO(n)} d\gamma \int\limits_{\mathbb{R}^{n-k}} \frac{F(\gamma(\mathbb{R}^k \oplus \mathbb{R}e_y))\,dy}{(1+|y|^2)^{(n+1)/2}}; \quad e_y = \frac{y+e_{n+1}}{|y+e_{n+1}|}. \end{aligned} \tag{3.9}$$

By (3.1) and (3.2), $\mathbb{R}^k \oplus \mathbb{R}e_y = \mu(\tau)$ for $\tau = (\mathbb{R}^k, y) \in G(n,k)$. Hence

$$I = \frac{\sigma_k}{\sigma_n}\int\limits_{SO(n)} d\gamma \int\limits_{\mathbb{R}^{n-k}} \frac{f(\gamma(\mathbb{R}^k + y))\,dy}{(1+|y|^2)^{(n+1)/2}} = \frac{\sigma_k}{\sigma_n}\int\limits_{G_{n,k}} d\xi \int\limits_{\xi^\perp} \frac{f(\xi,u)\,du}{(1+|u|^2)^{(n+1)/2}},$$

as was required. □

3.2. **Correspondence between Radon transforms.** Our next goal is to extend the correspondence (3.4) to Radon transforms and to dual Radon transforms. Let $G_{n+1,k+1}$ and $G_{n+1,k'+1}$ be a pair of (ordinary) Grassmann manifolds, $k' > k$. For a function $F(\tau_0)$ on $G_{n+1,k+1}$, we consider the Radon transform

$$(RF)(\zeta_0) = \int\limits_{\tau_0 \subset \zeta_0} F(\tau_0), \qquad \zeta_0 \in G_{n+1,k'+1}, \tag{3.10}$$

that integrates $F(\tau_0)$ over the set $G_{k+1}(\zeta_0)$ of all $(k+1)$-dimensional subspaces of ζ_0 against the canonical normalized measure on $G_{k+1}(\zeta_0)$. If g_{ζ_0} is a rotation such that $g_{\zeta_0}\mathbb{R}^{k'+1} = \zeta_0$, then (3.10) reads

$$(RF)(\zeta_0) = \int\limits_{G_{k'+1,k+1}} F(g_{\zeta_0}\tau_0)d\tau_0. \tag{3.11}$$

The dual Radon transform $(R^*\Phi)(\tau_0)$ of a function $\Phi(\zeta_0)$ on $G_{n+1,k'+1}$ integrates $\Phi(\zeta_0)$ over the set of all $(k'+1)$-dimensional subspaces ζ_0 containing the $(k+1)$-dimensional subspace τ_0, namely,

$$(R^*\Phi)(\tau_0)=\int\limits_{\zeta_0\supset\tau_0}\Phi(\zeta_0),\qquad \tau_0\in G_{n+1,k+1}. \tag{3.12}$$

Precise meaning of this integral is as follows. We denote by g_{τ_0} a rotation satisfying $g_{\tau_0}\mathbb{R}^{k+1}=\tau_0$, and let K_0 be the subgroup of rotations in the coordinate plane of $(\mathbb{R}^{k+1})^\perp$. Then (3.12) reads

$$(R^*\Phi)(\tau_0)=\int\limits_{K_0}\Phi(g_{\tau_0}\gamma\mathbb{R}^{k'+1})\,d\gamma. \tag{3.13}$$

The duality between R and R^* has the form

$$\int\limits_{G_{n+1,k'+1}}\Phi(\zeta_0)\,(RF)(\zeta_0)\,d\zeta_0=\int\limits_{G_{n+1,k+1}}(R^*\Phi)(\tau_0)\,F(\tau_0)\,d\tau_0; \tag{3.14}$$

see, e.g., Lemma 4.3 in [GR]. This equality holds provided that either integral exists in the Lebesgue sense.

By setting $\Phi=1$ or $F=1$ in (3.14), we get the following

Lemma 3.3. *The Radon transform R and its dual R^* are linear bounded operators from $L^1(G_{n+1,k+1})$ to $L^1(G_{n+1,k'+1})$, and from $L^1(G_{n+1,k'+1})$ to $L^1(G_{n+1,k+1})$, respectively.*

The following theorem essentially generalizes the statement of Theorem 2.1 from [Ku] related to totally geodesic Radon transforms on $\mathbb{R}^n$ and S^n (in our notation this is the case $k=0$).

Theorem 3.4. *Let $\tau\in G(n,k),\ \zeta\in G(n,k'),\ k'>k,$*

$$\rho_1(\tau)=(1+|\tau|^2)^{-(k'+1)/2},\qquad \rho_2(\zeta)=(1+|\zeta|^2)^{-(k+1)/2}, \tag{3.15}$$

$$\rho_3(\tau_0)=1/\rho_1(\mu^{-1}(\tau_0))=(\cos d(e_{n+1},\tau_0))^{-k'-1}, \tag{3.16}$$

$\tau_0\in G_{n+1,k+1}$. If $f(\tau)=F(\mu(\tau))$, then

$$(\rho_1 f)^\wedge(\zeta)=\frac{\sigma_{k'}}{\sigma_k}\rho_2(\zeta)(RF)(\mu(\zeta)), \tag{3.17}$$

$$\hat f(\zeta)=\frac{\sigma_{k'}}{\sigma_k}\rho_2(\zeta)(R\rho_3F)(\mu(\zeta)), \tag{3.18}$$

provided expressions in either side are finite. In particular, if $f\in L^1_{n-k'}(G(n,k))$, then $\rho_3F\in L^1(G_{n+1,k+1})$ (and vice versa), and both sides of (3.18) *are well defined.*

Proof. We recall the notation for coordinate planes

$$\mathbb{R}^k=\mathbb{R}e_1\oplus\ldots\oplus\mathbb{R}e_k,\quad \mathbb{R}^{k'}=\mathbb{R}e_1\oplus\ldots\oplus\mathbb{R}e_{k'},$$

$$\mathbb{R}^{k'-k}=\mathbb{R}e_{k+1}\oplus\ldots\oplus\mathbb{R}e_{k'}.$$

For $\zeta=(\eta,v)\in G(n,k')$, $\eta\in G_{n,k'}$, $v\in\eta^\perp$, let $\zeta_0=\mu(\zeta)\in G_{n+1,k'+1}$, and $\theta=d(e_{n+1},\zeta_0)$ be the corresponding geodesic distance. Then $\tan\theta=|v|=|\zeta|$. We choose a rotation $g\in SO(n)$ so that $g\mathbb{R}^{k'}=\eta$ and $ge_{k'+1}=v/|v|$. If

$$\zeta_\theta=\mathrm{span}(e_1,\ldots,e_{k'},e_{k'+1}\sin\theta+e_{n+1}\cos\theta)\in G_{n+1,k'+1},$$

then, clearly, $g\zeta_\theta = \zeta_0$. Changing variables, we have

$$(RF)(\zeta_0) = \int_{\tau_0 \subset \zeta_0} F(\tau_0) = \int_{\tau_0 \in G_{k+1}(\zeta_\theta)} F(g\tau_0). \tag{3.19}$$

If g_θ is a rotation in the plane $(e_{k'+1}, e_{n+1})$ such that

$$\zeta_\theta = g_\theta(\mathbb{R}^{k'} \oplus \mathbb{R}e_{n+1}), \quad g_\theta e_{n+1} = e_{k'+1}\sin\theta + e_{n+1}\cos\theta, \tag{3.20}$$

then (3.19) can be written as

$$(RF)(\zeta_0) = \int_\Omega h(\tau_0),$$

$$\Omega = \{\tau_0 : \tau_0 \in G_{k+1}(\mathbb{R}^{k'} \oplus \mathbb{R}e_{n+1})\}, \qquad h(\tau_0) = F(gg_\theta\tau_0). \tag{3.21}$$

Identifying τ_0 in the last integral with the k-geodesic $\tau_0 \cap S^n$ on $S^{k'} = S^n \cap (\mathbb{R}^{k'} \oplus \mathbb{R}e_{n+1})$, and making use of the formula (2.5) from [Ru2], we obtain

$$(RF)(\zeta_0) = c\int_0^{\pi/2} \sin^{k'-k-1}\omega \, \cos^k\omega \, d\omega \int_{SO(k')} h(\gamma g_\omega^{-1}\mathbb{R}^{k+1}) d\gamma, \tag{3.22}$$

where c, g_ω, and $\mathbb{R}^{k+1}$ have the same meaning as in (3.7) and $SO(k')$ is the group of rotations in the coordinate plane $\mathbb{R}^{k'}$. Since

$$g_\omega^{-1}\mathbb{R}^{k+1} = \mathbb{R}^k \oplus \mathbb{R}g_\omega^{-1}e_{k+1}, \qquad g_\omega^{-1}e_{k+1} = e_{k+1}\sin\omega + e_{n+1}\cos\omega,$$

the inner integral in (3.22) can be written as

$$\int_{SO(k')} h(\gamma(\mathbb{R}^k \oplus \mathbb{R}(e_{k+1}\sin\omega + e_{n+1}\cos\omega)))\, d\gamma$$

$$\text{(replace } \gamma \text{ by } \gamma\lambda,\ \lambda \in SO(k'-k), \text{ and integrate in } \lambda)$$

$$= \int_{SO(k')} d\gamma \int_{SO(k'-k)} h(\gamma(\mathbb{R}^k \oplus \mathbb{R}(\lambda e_{k+1}\sin\omega + e_{n+1}\cos\omega)))\, d\lambda.$$

We plug this in (3.22) and change the variable $\tan\omega = s$. We then get

$$\begin{aligned}(RF)(\zeta_0) &= \int_0^\infty \frac{s^{k'-k-1}\, ds}{(\sqrt{1+s^2})^{k'+1}} \\ &\times \int_{SO(k')} d\gamma \int_{SO(k'-k)} h(\gamma(\mathbb{R}^k \oplus \mathbb{R}(\frac{s\lambda e_{k+1}}{\sqrt{1+s^2}} + \frac{e_{n+1}}{\sqrt{1+s^2}})))\, d\lambda \\ &= \frac{\sigma_k}{\sigma_{k'}} \int_{SO(k')} d\gamma \int_{\mathbb{R}^{k'-k}} \frac{h(\gamma(\mathbb{R}^k \oplus \mathbb{R}e_y))\, dy}{(1+|y|^2)^{(k'+1)/2}}; \quad e_y = \frac{y+e_{n+1}}{|y+e_{n+1}|}.\end{aligned}$$

Owing to (3.21), $h(\tau_0) = F(gg_\theta\tau_0)$, and therefore

$$h(\gamma(\mathbb{R}^k \oplus \mathbb{R}e_y)) = F(gg_\theta\gamma(\mathbb{R}^k \oplus \mathbb{R}e_y)) = F(g\gamma(\mathbb{R}^k \oplus \mathbb{R}g_\theta e_y))$$

(rotations $\gamma \in SO(k')$ and g_θ in $(e_{k'+1}, e_{n+1})$ commute). By (3.20),

$$\begin{aligned} g_\theta e_y &= \frac{y + g_\theta e_{n+1}}{\sqrt{1+|y|^2}} = \frac{e_{k'+1}\sin\theta + e_{n+1}\cos\theta + y}{\sqrt{1+|y|^2}} \\ &= \frac{e_{k'+1}\tan\theta + z + e_{n+1}}{\sqrt{\tan^2\theta + |z|^2 + 1}}, \qquad z = y/\cos\theta. \end{aligned}$$

Combining these equalities, and setting $y = z\cos\theta$, we obtain

$$(RF)(\zeta_0) = \frac{\sigma_k}{\sigma_{k'}\cos^{k+1}\theta} \int\limits_{SO(k')} d\gamma \int\limits_{\mathbb{R}^{k'-k}} \frac{F(g\gamma\tau_{z,\theta})\, dz}{(\tan^2\theta + |z|^2 + 1)^{(k'+1)/2}},$$

$$\tau_{z,\theta} = \mathbb{R}^k \oplus \mathbb{R}e_{z,\theta}, \qquad e_{z,\theta} = \frac{e_{k'+1}\tan\theta + z + e_{n+1}}{|e_{k'+1}\tan\theta + z + e_{n+1}|}. \tag{3.23}$$

By (3.1) and (3.2),

$$\tau_{z,\theta} = \mu(\tau), \quad \tau = (\mathbb{R}^k, e_{k'+1}\tan\theta + z), \quad \tan\theta = |\zeta|.$$

Hence the correspondence (3.4) yields

$$F(g\gamma\tau_{z,\theta}) = f(g\gamma(\mathbb{R}^k + |\zeta|e_{k'+1} + z)). \tag{3.24}$$

Comparing (3.23) and (3.24) with (2.4), we conclude that

$$(RF)(\mu(\zeta)) = \frac{\sigma_k}{\sigma_{k'}}(1+|\zeta|^2)^{(k+1)/2}\hat{\psi}(\zeta), \qquad \psi(\tau) = \frac{f(\tau)}{(1+|\tau|^2)^{(k'+1)/2}}.$$

This coincides with (3.17). The equality (3.18) is a consequence of (3.17). Furthermore, by (3.5),

$$\begin{aligned} \int\limits_{G_{n+1,k+1}} (\rho_3 F)(\tau_0)\, d\tau_0 &= \int\limits_{G_{n+1,k+1}} \frac{F(\tau_0)\, d\tau_0}{\cos^{k'+1} d(e_{n+1}, \tau_0)} \\ &= \frac{\sigma_k}{\sigma_n} \int\limits_{G(n,k)} \frac{f(\tau)\, d\tau}{(1+|\tau|^2)^{(n-k')/2}}. \end{aligned}$$

This means that $f \in L^1_{n-k'}(G(n,k))$ if and only if $\rho_3 F \in L^1(G_{n+1,k+1})$. Hence, by Corollary 2.9 and Lemma 3.3, both sides of (3.18) are well defined for $f \in L^1_{n-k'}(G(n,k))$. The proof is complete. □

3.3. **Correspondence between dual Radon transforms.** The following statement generalizes Theorem 3.2(1) from [BCK] corresponding to the case of points in $\mathbb{R}^n$.

Theorem 3.5. *Let* $0 \le k < k' < n$,

$$\begin{aligned} &\tau \in G(n,k), \qquad \tau_0 = \mu(\tau) \in G_{n+1,k+1}, \\ &\zeta \in G(n,k'), \qquad \zeta_0 = \mu(\zeta) \in G_{n+1,k'+1}. \end{aligned}$$

We denote

$$\rho_4(\zeta) = (1+|\zeta|^2)^{(k-n)/2}, \qquad \rho_5(\tau) = (1+|\tau|^2)^{(k'-n)/2}, \tag{3.25}$$

$$\rho_6(\zeta_0) = 1/\rho_4(\mu^{-1}(\zeta_0)) = \cos^{k-n} d(e_{n+1}, \zeta_0), \tag{3.26}$$

and set $\varphi(\zeta) = \Phi(\mu(\zeta))$.

(i) *If* $\varphi \in L^1_{n+1}(G(n,k'))$, *i.e.*,

$$\int_{G(n,k')} \frac{|\varphi(\zeta)|\, d\zeta}{(1+|\zeta|^2)^{(n+1)/2}} < \infty, \tag{3.27}$$

then $\Phi \in L^1(G_{n+1,k'+1})$ *and*

$$(\rho_4\varphi)^\vee(\tau) = \rho_5(\tau)(R^*\Phi)(\mu(\tau)). \tag{3.28}$$

(ii) *If* $\varphi \in L^1_{k+1}(G(n,k'))$, *i.e.*,

$$\int_{G(n,k')} \frac{|\varphi(\zeta)|\, d\zeta}{(1+|\zeta|^2)^{(k+1)/2}} < \infty, \tag{3.29}$$

then $\rho_6\Phi \in L^1(G_{n+1,k'+1})$ *and*

$$\check{\varphi}(\tau) = \rho_5(\tau)(R^*\rho_6\Phi)(\mu(\tau)). \tag{3.30}$$

Proof. (i) It suffices to show that for any function $F(\tau_0) \in C^\infty(G_{n+1,k+1})$,

$$\int_{G_{n+1,k+1}} F(\tau_0)\,(R^*\Phi)(\tau_0)\, d\tau_0 = \int_{G_{n+1,k+1}} F(\tau_0)\,\psi(\tau_0)\, d\tau_0, \tag{3.31}$$

where $\psi(\mu(\tau)) = (\rho_4\varphi)^\vee(\tau)/\rho_5(\tau)$. Indeed, if

$$R^*\Phi \in L^1(G_{n+1,k+1}) \quad \text{and} \quad \psi(\tau_0) \in L^1(G_{n+1,k+1}), \tag{3.32}$$

then (3.31) implies (3.28) for almost all τ. By (3.5),

$$\int_{G(n,k')} \frac{\varphi(\zeta)\, d\zeta}{(1+|\zeta|^2)^{(n+1)/2}} = \frac{\sigma_n}{\sigma_{k'}} \int_{G_{n+1,k'+1}} \Phi(\zeta_0)\, d\zeta_0. \tag{3.33}$$

Hence $\Phi \in L^1(G_{n+1,k+1})$, and by Lemma 3.3, $R^*\Phi \in L^1(G_{n+1,k+1})$. Furthermore, (3.5), (3.25), and (2.18) yield

$$\begin{aligned} \int_{G_{n+1,k+1}} |\psi(\tau_0)|\, d\tau_0 \quad &\le \quad \frac{\sigma_k}{\sigma_n} \int_{G(n,k)} \frac{(\rho_4|\varphi|)^\vee(\tau)\, d\tau}{(1+|\tau|^2)^{(k'+1)/2}} \\ &= \quad \frac{\lambda_2\sigma_k}{\sigma_n} \int_{G(n,k')} \frac{|\varphi(\zeta)|\, d\zeta}{(1+|\zeta|^2)^{(n+1)/2}} < \infty. \end{aligned}$$

Thus (3.32) holds, and it remains to prove (3.31). By making use of duality (3.14) and (3.33), we obtain

$$\begin{aligned} I \quad &= \int_{G_{n+1,k+1}} F(\tau_0)\,(R^*\Phi)(\tau_0)\, d\tau_0 = \int_{G_{n+1,k'+1}} (\mathcal{R}F)(\zeta_0)\,\Phi(\zeta_0)\, d\zeta_0 \\ &= \quad \frac{\sigma_{k'}}{\sigma_n} \int_{G(n,k')} (\mathcal{R}F)(\mu(\zeta))\,\Phi(\mu(\zeta)) \frac{d\zeta}{(1+|\zeta|^2)^{(n+1)/2}}. \end{aligned}$$

Now we set $f(\tau) = F(\mu(\tau))$, $\varphi(\zeta) = \Phi(\mu(\zeta))$, and make use of (3.17). We get

$$I = \frac{\sigma_k}{\sigma_n} \int_{G(n,k')} (\rho_1 f)^\wedge(\zeta)\ \varphi(\zeta)\ \frac{d\zeta}{(1+|\zeta|^2)^{(n-k)/2}}.$$

Hence, by duality (2.7),

$$\begin{aligned} I &= \frac{\sigma_k}{\sigma_n} \int\limits_{G(n,k)} \Big[\frac{f(\tau)}{(1+|\tau|^2)^{(k'+1)/2}}\Big]\Big[\frac{\varphi(\zeta)}{(1+|\zeta|^2)^{(n-k)/2}}\Big]^\vee(\tau)\,d\tau \\ &\overset{(3.5)}{=} \int\limits_{G_{n+1,k+1}} \Big[\frac{F(\tau_0)}{(1+|\mu^{-1}(\tau_0)|^2)^{(k'-n)/2}}\Big]\Big[\frac{\varphi(\zeta)}{(1+|\zeta|^2)^{(n-k)/2}}\Big]^\vee(\mu^{-1}(\tau_0))\,d\tau_0 \\ &= \int\limits_{G_{n+1,k+1}} F(\tau_0)\,\psi(\tau_0)\,d\tau_0. \end{aligned}$$

(ii) The relation $\rho_6\Phi \in L^1(G_{n+1,k'+1})$ follows from (3.29) by (3.5):

$$\int\limits_{G_{n+1,k'+1}} (\rho_6\Phi)(\zeta_0)\,d\zeta_0 = \frac{\sigma_{k'}}{\sigma_n}\int\limits_{G(n,k')} \frac{\varphi(\zeta)\,d\zeta}{(1+|\zeta|^2)^{(k+1)/2}} < \infty. \tag{3.34}$$

Hence, the Radon transform in the right-hand side of (3.30) is well defined, and we can derive (3.30) from (3.28) by changing notation. □

4. Inversion of the Radon transform and the dual Radon transform

4.1. **Preliminaries.** Owing to the correspondence (3.18) and (3.30), and the obvious equality

$$(R^*\Phi)(\tau_0) = (R\Phi^\perp)(\tau_0^\perp), \qquad \Phi^\perp(\zeta_0^\perp) \overset{\text{def}}{=} \Phi(\zeta_0)$$

(see Lemma 4.3 below), inversion of the Radon transform and its dual on affine Grassmannians reduces to the similar problem on ordinary Grassmannians [GR]. We review some facts from [GR] and present them in our notation. Let $\mathcal{P}_{k+1}$ be the cone of positive definite, symmetric $(k+1)\times(k+1)$ matrices $r=(r_{i,j})$. The Siegel Gamma function associated to $\mathcal{P}_{k+1}$ is defined by

$$\Gamma_{k+1}(\alpha) = \int\limits_{\mathcal{P}_{k+1}} e^{-\text{tr}(r)}\,\det(r)^{\alpha-d}dr, \qquad \text{tr}(r) = \text{trace of } r, \tag{4.1}$$

$d=(k+2)/2$, $dr = \prod_{i\le j} dr_{i,j}$. This integral converges for $Re\,\alpha > d-1$, and represents the product of the usual Γ-functions:

$$\Gamma_{k+1}(\alpha) = \pi^{k(k+1)/4}\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})\dots\Gamma(\alpha-\frac{k}{2}). \tag{4.2}$$

The Gårding-Gindikin fractional integral of a function f on $\mathcal{P}_{k+1}$ is defined by

$$(I^\alpha_+ f)(r) = \frac{1}{\Gamma_{k+1}(\alpha)}\int\limits_0^r f(s)\,\det(r-s)^{\alpha-d}ds, \quad Re\,\alpha > d-1, \tag{4.3}$$

where $\int_0^r$ denotes integration over the set $\{s : s\in\mathcal{P}_{k+1},\ r-s\in\mathcal{P}_{k+1}\}$. We define a differential operator (in the r-variable)

$$D_+ = \det\left(\eta_{i,j}\,\frac{\partial}{\partial r_{i,j}}\right), \quad \eta_{i,j} = \begin{cases} 1 & \text{if } i=j, \\ 1/2 & \text{if } i\ne j, \end{cases} \tag{4.4}$$

so that

$$D^m_+ I^\alpha_+ f = I^{\alpha-m}_+ f, \qquad m\in\mathbb{N}, \quad Re\,\alpha > m+d-1,$$

if f is good enough. Otherwise, differentiation in this equality is understood in the sense of distributions.

Let $\tau_0 \in G_{n+1,k+1},\ \zeta_0 \in G_{n+1,k'+1},\ 0 \le k < k' \le n-1$. Given a function $\Psi(\zeta_0)$ on $G_{n+1,k'+1}$, we introduce the mean value operator

$$(M_r^*\Psi)(\tau_0) = \int\limits_{\{\zeta_0 :\, \mathrm{Cos}^2(\zeta_0,\tau_0)=r\}} \Psi(\zeta_0)\, dm(\zeta_0), \quad r \in \mathcal{P}_{k+1}, \tag{4.5}$$

where $\mathrm{Cos}^2(\zeta_0,\tau_0) \overset{\text{def}}{=} y'\mathrm{Pr}_{\zeta_0} y, \quad y = [y_1,\dots,y_{k+1}]$, is a matrix whose columns form an orthonormal basis of τ_0. Detailed explanations related to the integral (4.5) and the matrix-valued cosine function are given in [GR, Section 3]. The operator (4.5) is a matrix generalization of the averaging operator of Helgason (cf. formula (35) in [H2, p. 96]) in his inversion procedure for the totally geodesic Radon transform on the sphere.

Theorem 4.1. *Let $\Phi(\tau_0) \in L^1(G_{n+1,k+1})$. Suppose that*

$$\Psi(\zeta_0) = (\mathcal{R}\Phi)(\zeta_0), \quad \zeta_0 \in G_{n+1,k'+1}, \quad 0 \le k < k' \le n-1,$$

and denote

$$\alpha = \frac{k'-k}{2}, \quad \tilde{\Psi}_{\tau_0}(r) = \det(r)^{\alpha-1/2}(M_r^*\Psi)(\tau_0), \quad c = \frac{\Gamma_{k+1}((k+1)/2)}{\Gamma_{k+1}((k'+1)/2)}.$$

The operator R is injective if and only if

$$k + k' \le n-1. \tag{4.6}$$

Under this condition, $\Phi(\tau_0)$ can be recovered by the formula $\Phi = R^{-1}\Psi$, where

$$(R^{-1}\Psi)(\tau_0) = c \overset{(L^1)}{\lim_{r \to I_{k+1}}} (D_+^m I_+^{m-\alpha} \tilde{\Psi}_{\tau_0})(r), \quad \forall m > k'/2,\ m \in \mathbb{N}, \tag{4.7}$$

I_{k+1}, is the identity $(k+1) \times (k+1)$ matrix, and differentiation is understood in the sense of distributions. In particular, for $k'-k = 2\ell,\ \ell \in \mathbb{N}$,

$$(R^{-1}\Psi)(\tau_0) = c \overset{(L^1)}{\lim_{r \to I_{k+1}}} (D_+^\ell \tilde{\Psi}_{\tau_0})(r). \tag{4.8}$$

If Φ is a continuous function on $G_{n+1,k+1}$, then the limits in (4.7) *and* (4.8) *can be treated in the* sup*-norm.*

This theorem was proved in [GR, Theorem 1.2]. The necessity of the injectivity condition $k + k' \le n-1$ was known before; see references in [GR].

4.2. **Inversion of the Radon transform.** Let $\tau \in G(n,k)$. According to Corollary 2.9, we assume $f \in L^1_{n-k'}(G(n,k))$, i.e.,

$$\int\limits_{G(n,k)} \frac{|f(\tau)|\, d\tau}{(1+|\tau|)^{n-k'}} < \infty,$$

and set $\tau_0 = \mu(\tau),\ f(\tau) = F(\tau_0)$; see (3.1), (3.4). By Theorem 3.4,

$$\hat{f}(\zeta) = \frac{\sigma_{k'}}{\sigma_k}\rho_2(\zeta)(R\rho_3 F)(\mu(\zeta)), \tag{4.9}$$

where $\zeta \in G(n,k'),\ \rho_2(\zeta) = (1+|\zeta|^2)^{-(k+1)/2}$,

$$\rho_3(\tau_0) = (\cos d(e_{n+1},\tau_0))^{-k'-1} = (1+|\tau|^2)^{(k'+1)/2}, \quad \rho_3 F \in L^1(G_{n+1,k+1}).$$

In particular, one can assume

$$f \in L^p(G(n,k)), \qquad 1 \le p < (n-k)/(k'-k);$$

see Remark 2.10.

Theorem 4.1 implies the following result for Radon transforms on affine Grassmannians.

Theorem 4.2. *Let $\tau \in G(n,k)$, $\zeta \in G(n,k')$, $0 \le k < k' \le n-1$. For $f \in L^1_{n-k'}(G(n,k))$, the Radon transform $f(\tau) \to \hat{f}(\zeta)$ is injective if and only if $k+k' \le n-1$. Under this condition, the function $f(\tau)$ can be recovered by the formula*

$$f(\tau) = (1+|\tau|^2)^{-(k'+1)/2}(R^{-1}\Psi)(\mu(\tau)), \tag{4.10}$$

where

$$\Psi(\mu(\zeta)) = \frac{\sigma_k}{\sigma_{k'}}(1+|\zeta|^2)^{(k+1)/2}\hat{f}(\zeta),$$

and the operator R^{-1} is defined by (4.7), (4.8). If $f(\tau)$ is a continuous function satisfying $f(\tau) = O(|\tau|^{-\lambda})$, $\lambda > k'-k$, then the limit in (4.7) and (4.8) can be understood in the sup*-norm on any compact subset of $G(n,k)$. If $\lambda > k'+1$, this limit is uniform on $G(n,k)$.*

The inversion formula (4.10) is obvious thanks to (4.9). Concerning the case of a continuous function $f(\tau) = O(|\tau|^{-\lambda})$, we note that $F(\tau_0) \equiv f(\mu^{-1}(\tau_0)) = O(\cos^\lambda\theta)$, $\theta = d(e_{n+1}, \tau_0)$. The function $(\rho_3 F)(\tau_0)$ in (4.9) is continuous on the set $G_{n+1,k+1} \setminus \mathcal{E}$,

$$\mathcal{E} = \{\tau_0 : \, d(e_{n+1}, \tau_0) = \frac{\pi}{2}\} = G^0_{n+1,k+1}$$

(see (3.3)). The "equator" $\mathcal{E}$ of $G_{n+1,k+1}$ corresponds to $|\tau| = \infty$ on $G(n,k)$. Near it, $(\rho_3 F)(\tau_0)$ may have an integrable singularity. This explains why the last statement of Theorem 4.2 holds globally only if $\lambda > k'+1$, when $(\rho_3 F)(\tau_0) = O(\cos^{\lambda-k'-1}\theta) = o(1)$ as $\theta \to \frac{\pi}{2}$.

4.3. **Inversion of the dual Radon transform.** We start with the following

Lemma 4.3. *Let $0 \le k < k' < n$, $\tau_0 \in G_{n+1,k+1}$, $\zeta_0 \in G_{n+1,k'+1}$,*

$$t_0 = \tau_0^\perp \in G_{n+1,n-k}, \qquad z_0 = \zeta_0^\perp \in G_{n+1,n-k'}. \tag{4.11}$$

Given functions $F(\tau_0)$ on $G_{n+1,k+1}$, and $\Phi(\zeta_0)$ on $G_{n+1,k'+1}$, we denote

$$F^\perp(t_0) = F(t_0^\perp), \qquad \Phi^\perp(z_0) = \Phi(z_0^\perp). \tag{4.12}$$

The following relations hold:

$$\int\limits_{G_{n+1,n-k}} F^\perp(t_0)\,dt_0 = \int\limits_{G_{n+1,k+1}} F(\tau_0)\,d\tau_0, \tag{4.13}$$

$$\int\limits_{G_{n+1,n-k'}} \Phi^\perp(z_0)\,dz_0 = \int\limits_{G_{n+1,k'+1}} \Phi(\zeta_0)\,d\zeta_0, \tag{4.14}$$

$$(R\Phi^\perp)(t_0) = (R^*\Phi)(\tau_0), \qquad (RF)(\zeta_0) = (R^*F^\perp)(z_0). \tag{4.15}$$

Proof. For (4.13), we have

$$\int\limits_{G_{n+1,k+1}} F(\tau_0)\, d\tau_0 = \int\limits_{SO(n+1)} F(g\mathbb{R}^{k+1})dg = \int\limits_{SO(n+1)} F^{\perp}(g(\mathbb{R}^{k+1})^{\perp})dg = \int\limits_{G_{n+1,n-k}} F^{\perp}(t_0)\, dt_0.$$

Furthermore, if K_0 is the subgroup of rotations in the coordinate plane $(\mathbb{R}^{k+1})^{\perp}$, then, by (3.13),

$$(R^*\Phi)(\tau_0) = \int\limits_{K_0} \Phi(g_{\tau_0}\gamma\mathbb{R}^{k'+1})\, d\gamma,$$

where $g_{\tau_0} \in SO(n+1),\ g_{\tau_0}\mathbb{R}^{k+1} = \tau_0$, i.e., $g_{\tau_0}(\mathbb{R}^{k+1})^{\perp} = t_0$. The last integral reads

$$\int\limits_{K_0} \Phi^{\perp}(g_{\tau_0}\gamma(\mathbb{R}^{k'+1})^{\perp})\, d\gamma = \int\limits_{G_{n-k'}((\mathbb{R}^{k+1})^{\perp})} \Phi^{\perp}(g_{\tau_0}\lambda)d\lambda = (R\Phi^{\perp})(t_0).$$

The proof of (4.14) and the second equality in (4.15) is similar. □

The inversion procedure for the dual Radon transform is as follows. We suppose that $\varphi \in L^1_{k+1}(G(n,k'))$, i.e.,

$$N_{\varphi} \equiv \int\limits_{G(n,k')} \frac{|\varphi(\zeta)|\, d\zeta}{(1+|\zeta|)^{k+1}} < \infty. \tag{4.16}$$

Note that any function $\varphi \in L^p(G(n,k')),\ 1 \le p \le \infty$, satisfies (4.16) provided $k + k' \ge n-1$. Furthermore, any continuous function $\varphi(\zeta)$ such that

$$\varphi(\zeta) = O(|\zeta|^{\lambda}), \qquad \lambda < k' + k + 1 - n, \tag{4.17}$$

satisfies (4.16). Indeed, in the first case, N_{φ} is dominated by $A\,||\varphi||_p$, where

$$A^{p'} = \sigma_{n-k'-1} \int\limits_0^{\infty} \frac{r^{n-k'-1}}{(1+r)^{(k+1)p'}}\, dr < \infty.$$

In the second case,

$$N_{\varphi} \le c \int\limits_0^{\infty} \frac{r^{n-k'+\lambda-1}}{(1+r)^{k+1}}\, dr < \infty.$$

Let $\tau \in G(n,k),\ \zeta \in G(n,k')$. We set $\varphi(\zeta) = \Phi(\mu(\zeta))$,

$$\tau_0 = \mu(\tau) \in G_{n+1,k+1}, \qquad \zeta_0 = \mu(\zeta) \in G_{n+1,k'+1}, \tag{4.18}$$

$$t_0 = \tau_0^{\perp} \stackrel{\text{def}}{=} \mu_{\perp}(\tau) \in G_{n+1,n-k}, \quad z_0 = \zeta_0^{\perp} \stackrel{\text{def}}{=} \mu_{\perp}(\zeta) \in G_{n+1,n-k'}. \tag{4.19}$$

By Theorem 3.5 and (4.15),

$$\rho_5^{-1}(\tau)\check{\varphi}(\tau) = (R^*\rho_6\Phi)(\mu(\tau)) = (Rh)(t_0), \tag{4.20}$$

where $\rho_5(\tau) = (1+|\tau|^2)^{(k'-n)/2}$,

$$\begin{aligned} \rho_6(\zeta_0) = \cos^{k-n} d(e_{n+1},\zeta_0) &= (1+|\zeta|^2)^{(n-k)/2}, \\ h(z_0) &= (\rho_6\Phi)^{\perp}(z_0) = \rho_6(z_0^{\perp})\Phi(z_0^{\perp}) \\ &= \rho_6(\zeta_0)\Phi(\zeta_0) = (1+|\zeta|^2)^{(n-k)/2}\varphi(\zeta). \end{aligned} \tag{4.21}$$

Since $\rho_6\Phi \in L^1(G_{n+1,k'+1})$, then $h \in L^1(G_{n+1,n-k'})$, and the Radon transform in (4.20) can be inverted by Theorem 4.1 adapted to the corresponding pair of Grassmannians. As a result we obtain the following

Theorem 4.4. *Let $\tau \in G(n,k)$, $\quad \zeta \in G(n,k')$, $\quad 0 \le k < k' \le n-1$. For $\varphi \in L^1_{k+1}(G(n,k'))$, the dual Radon transform $\varphi(\zeta) \to \check{\varphi}(\tau)$ is injective if and only if $k+k' \ge n-1$. Under this condition, the function $\varphi(\zeta)$ can be recovered by the formula*

$$\varphi(\zeta) = (1+|\zeta|^2)^{(k-n)/2}(R^{-1}\Psi_\perp)(\mu_\perp(\zeta)), \tag{4.22}$$

where

$$\Psi_\perp(\mu_\perp(\tau)) = (1+|\tau|^2)^{(n-k')/2}\check{\varphi}(\tau).$$

The operator R^{-1} is defined by the formulas (4.7) *and* (4.8) *in which dimensions $k+1$ and $k'+1$ are replaced by $n-k$ and $n-k'$, respectively. If $\varphi(\zeta)$ is a continuous function satisfying $\varphi(\zeta) = O(|\zeta|^\lambda)$, $\lambda < k'+k+1-n$, then the limit in the updated formulas* (4.7) *and* (4.8) *can be understood in the* sup*-norm on any compact subset of $G(n,k')$. If $\lambda < k-n$, this limit is uniform on the whole manifold $G(n,k')$.*

For a continuous function $\varphi(\zeta) = O(|\zeta|^\lambda)$, as in the previous theorem, we note that $\Phi(\zeta_0) \equiv \varphi(\mu^{-1}(\zeta_0)) = O(\cos^{-\lambda}\theta)$, $\theta = d(e_{n+1},\zeta_0)$. The function $(\rho_6\Phi)(\zeta_0)$ in (4.20) is continuous on the set $G_{n+1,k'+1} \setminus \mathcal{E}'$, where the "equator" $\mathcal{E}' = \{\zeta_0 : d(e_{n+1},\zeta_0) = \frac{\pi}{2}\}$ of $G_{n+1,k'+1}$ corresponds to $|\zeta| = \infty$ on $G(n,k')$. Near it, $(\rho_6\Phi)(\zeta_0)$ may have an integrable singularity that disappears if $\lambda < k-n$, when $(\rho_6\Phi)(\zeta_0) = O(\cos^{k-n-\lambda}\theta) = o(1)$ as $\theta \to \frac{\pi}{2}$.

Theorems 4.2 and 4.4 imply the following interesting corollary.

Corollary 4.5. *Let $0 \le k < k' \le n-1$. The Radon transform $f(\tau) \to \hat{f}(\zeta)$ defined on $f \in L^1_{n-k'}(G(n,k))$ and acting from $G(n,k)$ to $G(n,k')$, and the dual Radon transform $\varphi(\zeta) \to \check{\varphi}(\tau)$ defined on $\varphi \in L^1_{k+1}(G(n,k'))$ and acting from $G(n,k')$ to $G(n,k)$, are injective simultaneously if and only if $k+k' = n-1$.*

5. Quasi-orthogonal inversion transformation on affine Grassmannians

In Section 3 we established correspondence between (a) the Radon transform $\hat{f}$ and the dual Radon transform $\check{\varphi}$ on affine Grassmannians, and (b) the similar transforms on ordinary Grassmannians. Since the latter are orthogonally connected by Lemma 4.3, it is natural to find a direct representation of $\check{\varphi}$ as a Radon transform of the "$\wedge$"-type. As we shall see, such a representation realizes by generalization of the classical map

$$x \to -\frac{x}{|x|^2}, \qquad x \in \mathbb{R}^n \setminus \{0\}$$

(inversion with respect to the unit sphere combined with reflection).

Definition 5.1. Let $\tau \in G(n,k)$ be a k-plane in $\mathbb{R}^n$ not passing through the origin and parameterized by $\tau = (\xi, u)$, $\xi \in G_{n,k}$, $u \in \xi^\perp$, $u \ne 0$. Let $\operatorname{lin}\tau \in G_{n,k+1}$ be the linear hull of τ (the smallest linear subspace containing τ), and let $(\operatorname{lin}\tau)^\perp \in G_{n,n-k-1}$ be the orthogonal complement of $\operatorname{lin}\tau$. We consider the $(n-k-1)$-dimensional plane $t = (\iota, w)$ defined by

$$\iota = (\operatorname{lin}\tau)^\perp = (\xi \oplus \mathbb{R}(u/|u|))^\perp \in G_{n,n-k-1}, \qquad w = -\frac{u}{|u|^2} \in \iota^\perp. \tag{5.1}$$

The mapping

$$G(n,k) \ni \tau \xrightarrow{\ \nu\ } t = \nu(\tau) \in G(n, n-k-1) \tag{5.2}$$

will be called a *quasi-orthogonal inversion transformation* from $G(n,k)$ to $G(n,n-k-1)$.

Clearly, $\nu(\nu(\tau)) = \tau$.

Example 5.2. If $k = 0$ and $\tau = x \in \mathbb{R}^n \setminus \{0\}$, then $\nu(x)$ is the hyperplane orthogonal to the vector x and passing through the point $-x/|x|^2$.

The following lemma motivates our definition.

Lemma 5.3. *Let*

$$\tau \in G(n,k) \setminus G_{n,k}, \qquad \tau_0 = \mu(\tau) \in G_{n+1,k+1}, \tag{5.3}$$

$$t_0 = \tau_0^\perp \in G_{n+1,n-k} \qquad t = \mu^{-1}(t_0) \in G(n, n-k-1). \tag{5.4}$$

If ν is a quasi-orthogonal inversion transformation from $G(n,k)$ to $G(n,n-k-1)$, then the following diagram is commutative:

$$\begin{array}{ccc} \tau & \xrightarrow{\ \mu\ } & \tau_0 \\ {\scriptstyle\nu}\big\downarrow & & \big\downarrow{\scriptstyle\perp} \\ t & \xleftarrow[\mu^{-1}]{} & t_0 \end{array}$$

In particular,

$$\nu^{-1}(t) = \mu^{-1}(\mu(t)^\perp). \tag{5.5}$$

Proof. For $\tau = (\xi, u)$, let $u = r\omega,\ r > 0,\ \omega = u/|u| \in S^{n-1} \cap \xi^\perp$. Then

$$\tau_0 = \mu(\tau) = \text{lin}\Big(\xi, \frac{u + e_{n+1}}{|u + e_{n+1}|}\Big) = g_\xi \ell,$$

where $g_\xi \in SO(n), \quad g_\xi \mathbb{R}^k = \xi, \quad g_\xi e_{k+1} = \omega,$

$$\ell = \text{lin}\Big(\mathbb{R}^k, \frac{re_{k+1} + e_{n+1}}{\sqrt{1+r^2}}\Big) = \text{span}(e_1, \dots, e_k,\ e_{k+1}\sin\theta + e_{n+1}\cos\theta),$$

$r = \tan\theta, \quad \theta = d(e_{n+1}, \tau_0)$. Hence $t_0 = \tau_0^\perp = g_\xi \ell^\perp$, where

$$\begin{aligned} \ell^\perp &= \text{span}(e_{k+2}, \dots, e_n,\ -e_{k+1}\cos\theta + e_{n+1}\sin\theta) \\ &= \text{lin}\Big((\mathbb{R}^{k+1})^\perp,\ \frac{(-1/r)e_{k+1} + e_{n+1}}{\sqrt{1+(1/r)^2}}\Big). \end{aligned}$$

This gives $t = \mu^{-1}(t_0) = (\iota, w)$, where

$$\begin{aligned} \iota &= g_\xi(\mathbb{R}^{k+1})^\perp = (g_\xi \mathbb{R}^{k+1})^\perp = (\xi \oplus \mathbb{R}\omega)^\perp = (\text{lin } \tau)^\perp, \\ w &= -\frac{1}{r} g_\xi e_{k+1} = -\frac{u}{|u|^2}. \end{aligned}$$

□

We introduce the following new notion.

Definition 5.4. Let $\mathcal{R} : f(\tau) \to \hat{f}(\zeta)$ be the Radon transform, acting from $G(n,k)$ to $G(n,k'),\ k' > k$. The Radon transform $\mathfrak{R} : \mathfrak{f}(t) \to \hat{\mathfrak{f}}(z)$, acting from $G(n, n-k'-1)$ to $G(n, n-k-1)$, will be called *quasi-orthogonal* to $\mathcal{R}$.

Theorem 5.5. *Let* $\varphi(\zeta)$ *be a function on* $G(n,k')$, *and let* $z=\nu(\zeta)$ *be the quasi-orthogonal inversion transformation, acting from* $G(n,k')$ *to* $G(n,n-k'-1)$. *We define a "Kelvin-type" transformation*

$$(K\varphi)(z)=|z|^{k-n}\varphi(\nu^{-1}(z)),\qquad z\in G(n,n-k'-1). \tag{5.6}$$

(i) *The following relation holds:*

$$\int\limits_{G(n,k')}\frac{\varphi(\zeta)\,d\zeta}{(1+|\zeta|^2)^{(k+1)/2}}=\frac{\sigma_{n-k'-1}}{\sigma_{k'}}\int\limits_{G(n,n-k'-1)}\frac{(K\varphi)(z)\,dz}{(1+|z|^2)^{(k+1)/2}}, \tag{5.7}$$

and, therefore,

$$\varphi\in L^1_{k+1}(G(n,k'))\Longleftrightarrow K\varphi\in L^1_{k+1}(G(n,n-k'-1)). \tag{5.8}$$

(ii) *If* $\varphi\in L^1_{k+1}(G(n,k'))$, *then*

$$\check{\varphi}(\tau)=c\,|\tau|^{k'-n}(K\varphi)^\wedge(\nu(\tau)),\qquad c=\frac{\sigma_{n-k'-1}}{\sigma_{n-k-1}}, \tag{5.9}$$

where $(K\varphi)^\wedge(t)$ *denotes the quasi-orthogonal Radon transform of* $(K\varphi)(z)$ *which acts from* $G(n,n-k'-1)$ *to* $G(n,n-k-1)$ *and is finite for almost all* $t\in G(n,n-k-1)$.

Proof. (i) We denote by I the left-hand side of (5.7). By Lemma 3.2, assuming $\varphi(\zeta)=\Phi(\mu(\zeta))=\Phi(\zeta_0)$, we have

$$I=\frac{\sigma_n}{\sigma_{k'}}\int\limits_{G_{n+1,k'+1}}\frac{\Phi(\zeta_0)\,d\zeta_0}{\cos^{n-k}d(e_{n+1},\zeta_0)}=\frac{\sigma_n}{\sigma_{k'}}\int\limits_{G_{n+1,k'+1}}\rho_6(\zeta_0)\,\Phi(\zeta_0)\,d\zeta_0,$$

where $\rho_6(\zeta_0)=\cos^{k-n}d(e_{n+1},\zeta_0)=(1+|\zeta|^2)^{(n-k)/2}$; see (3.26). Let

$$h(z_0)=\rho_6(z_0^\perp)\Phi(z_0^\perp),\quad z_0^\perp=\zeta_0,\quad z_0\in G_{n+1,n-k'}. \tag{5.10}$$

Then, by Lemmas 4.3 and 3.2,

$$I=\frac{\sigma_n}{\sigma_{k'}}\int\limits_{G_{n+1,n-k'}}h(z_0)\,dz_0=\frac{\sigma_{n-k'-1}}{\sigma_{k'}}\int\limits_{G(n,n-k'-1)}\frac{h(\mu(z))\,dz}{(1+|z|^2)^{(n+1)/2}}.$$

It remains to note that $|z|=|\zeta|^{-1}$, and therefore

$$\begin{aligned}\frac{h(\mu(z))}{(1+|z|^2)^{(n+1)/2}}&=\frac{(1+|\zeta|^2)^{(n-k)/2}\Phi(\mu(z)^\perp)}{(1+|z|^2)^{(n+1)/2}}=\frac{|z|^{k-n}\Phi(\mu(z)^\perp)}{(1+|z|^2)^{(k+1)/2}}\\&=\frac{|z|^{k-n}\varphi(\nu^{-1}(z))}{(1+|z|^2)^{(k+1)/2}}=\frac{(K\varphi)(z)\,dz}{(1+|z|^2)^{(k+1)/2}}.\end{aligned}$$

(ii) Let us prove (5.9). By Corollary 2.9, the Radon transform $(K\varphi)^\wedge(t)$ is finite for almost all $t\in G(n,n-k-1)$. Furthermore, by (3.30), (4.15), and (5.10),

$$\rho_5^{-1}(\tau)\check{\varphi}(\tau)=(R^*\rho_6\Phi)(\tau_0)=(Rh)(t_0)=(Rh)(\mu(t)), \tag{5.11}$$

where $\rho_5(\tau)=(1+|\tau|^2)^{(k'-n)/2}$. By (3.17),

$$(Rh)(\mu(t))=c\,\tilde{\rho}_2(t)(\tilde{\rho}_1\tilde{f})^\wedge(t),\qquad c=\frac{\sigma_{n-k'-1}}{\sigma_{n-k-1}}, \tag{5.12}$$

where $\tilde{f}(z)=h(\mu(z))$,

$$\tilde{\rho}_1(z)=(1+|z|^2)^{(k-n)/2},\qquad \tilde{\rho}_2(t)=(1+|t|^2)^{(n-k')/2}. \tag{5.13}$$

Since $|z| = |\zeta|^{-1}$ and $|t| = |\tau|^{-1}$, (5.11)–(5.13) yield

$$\check{\varphi}(\tau) = c\,|\tau|^{k'-n}[|z|^{k-n}\varphi(\nu^{-1}(z))]^{\wedge}(\nu(\tau)) = c\,|\tau|^{k'-n}(K\varphi)^{\wedge}(\nu(\tau)).$$

□

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